JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 30, 313-329 (1970)

On an Expansion in Cauchy Exponential Series

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Submitted by N. G. de Bruijn

If $f \in L(0, 1)$ and $Q(z)$ is a meromorphic function with poles z_ν, then the Cauchy Exponential Series (C.E.S.) of f with respect to $Q(z)$ is $\Sigma c_\nu e^{z_\nu x}$, where

$$c_\nu e^{z_\nu x} = \operatorname{res}_{z_\nu} Q(z) \int_0^1 f(t)\, e^{z(x-t)}\, dt. \tag{1}$$

Various classes of C.E.S. have already been investigated (e.g., [1], [4], [7]). This paper investigates the case when

$$Q(z) = \frac{\int_0^1 e^{zt} \varphi(t)\, dt}{2 \int_0^1 \sinh zt\, \varphi(t)\, dt}, \tag{2}$$

where φ is an absolutely continuous (AC) function.

We show how the behaviour of the C.E.S. is related to that of a Fourier series under a variety of conditions on f and φ; a type of Bessel inequality is proved, and the special case when f is taken to be φ, which yields an interesting identity for the coefficients, is also considered.

1. INTRODUCTION

We suppose that φ is AC in $[0, 1]$, and we take $\varphi(1) = 1$. For convenience, we shall further suppose that $\varphi(0) = 0$. This restriction can be removed without difficulty (see Section 4).

We have from (2),

$$Q(z) = \frac{e^z - \int_0^1 e^{zt} \varphi'(t)\, dt}{2\left(\cosh z - \int_0^1 \cosh zt\, \varphi'(t)\, dt\right)}. \tag{3}$$

If we write $z = \xi + i\eta$, then, in the strip $|\xi| < c$,

$$\int_0^1 e^{zt} \varphi'(t)\, dt = o(1), \qquad \int_0^1 e^{-zt} \varphi'(t)\, dt = o(1),$$

as $|\eta| \to \infty$; so the large poles of $Q(z)$ approach those of sech z, that is, the points $(\nu + \frac{1}{2})\pi i$, where ν is large and integral.

Denote by C_p the circle $\rho = |z| = p\pi$, and by C_p^+, C_p^- the parts of C_p lying in the right and left halfplanes, respectively. For large p, it follows that on C_p,

$$\left| \cosh z - \int_0^1 \cosh zt\, \varphi'(t)\, dt \right| > K |\cosh z|, \tag{4}$$

where K is a constant.

The numbers $\{z_\nu\}$, which are the poles of $Q(z)$, are included among the zeros of

$$\int_0^1 \sinh zt\, \varphi(t)\, dt. \tag{5}$$

There are at most a finite number of nonsimple poles of $Q(z)$, since the large poles approach $(\nu + \frac{1}{2})\pi i$. Moreover, the number of zeros of (5) (counting multiplicity) inside C_p is precisely one less than the number of zeros of

$$\begin{aligned} A(z) &= z \int_0^1 \sinh zt\, \varphi(t)\, dt \\ &= \cosh z - \int_0^1 \cosh zt\, \varphi'(t)\, dt, \end{aligned}$$

in C_p. For large p, the number of zeros of $A(z)$ in C_p is the same as the number of zeros of cosh z in C_p, viz. $2p$. So, counting multiplicity, $Q(z)$ has at most $2p - 1$ poles in C_p.

If they are distinct, denote them by z_ν ($\nu = 0, \pm 1, \ldots, \pm p - 1$), choosing the notation so that

$$\begin{aligned} z_\nu &= (\nu - \tfrac{1}{2})\pi i + o(1), \\ z_{-\nu} &= -(\nu - \tfrac{1}{2})\pi i + o(1), \end{aligned} \tag{6}$$

for large positive ν.

We know that for large $\nu > 0$, there is a simple pole near $(\nu - \frac{1}{2})\pi i$, and a simple pole near $-(\nu - \frac{1}{2})\pi i$, and that these are the only large poles. So if p is large, the number of poles inside C_p of $Q(z)$, counting multiplicity, is $2p - 1 - k$, where k is a constant. Nevertheless, for large ν, we may still write the relations (6), provided that we make the convention that for certain low values of ν, $c_\nu = 0$.

The purpose of these remarks is to allow us to say that $\Sigma_{|z_\nu|<p\pi}$ contains $2p-1$ terms, just as $\Sigma_{\nu=-p+1}^{\nu=p-1}$ does.

If $z=0$ is a pole of $Q(z)$, denote it by z_0. If $z=0$ is not a pole, then denote it by z_0 and put $c_0=0$. Then,

$$\left(\frac{z_\nu}{p\pi}\right)^2 = -\left(\frac{\nu}{p}\right)^2 + O\left(\frac{\nu}{p^2}\right). \tag{7}$$

Now, write

$$\begin{aligned} S_p(x) &= \frac{1}{2\pi i}\int_{C_p} Q(z)\,dz \int_0^1 f(t)\,e^{z(x-t)}\,dt \\ &= \Sigma_{|z_\nu|<p\pi}\, c_\nu e^{z_\nu x} \end{aligned} \tag{8}$$

by (1).

We may also write

$$Q(z) = \frac{e^z}{2\cosh z} + R(z) = Q_1(z) + R(z).$$

Then,

$$\begin{aligned} S_p(x) &= \frac{1}{2\pi i}\int_{C_p} Q_1(z)\,dz\int_0^1 f(t)\,e^{z(x-t)}\,dt + \frac{1}{2\pi i}\int_{C_p} R(z)\,dz\int_0^1 f(t)\,e^{z(x-t)}\,dt \\ &= \sigma_p(x) + J_p(x), \end{aligned} \tag{9}$$

say.

We investigate the C.E.S. by considering the behaviour of $\sigma_p(x)$ and $J_p(x)$ separately. We first prove

LEMMA 1. *Let $f\in L(0,1)$. Denote by $t_p(x)$ the* p*th partial sum of the Fourier series* (F.s.) *at x of the function $g(\tau)\,e^{-\pi i\tau}$, where g is given by*

$$g(\tau) = \begin{cases} f(2\tau) & 0\leqslant\tau\leqslant\frac{1}{2}, \\ 0 & \frac{1}{2}<\tau\leqslant 1. \end{cases}$$

Then, $\sigma_p(x) - e^{\pi i x/2}t_p(x/2)\to 0$ as $p\to\infty$, uniformly in $[0, 1]$.

Proof. We have, by (9),

$$\begin{aligned} \sigma_p(x) &= \frac{1}{2\pi i}\int_{C_p}\frac{e^z}{e^z+e^{-z}}\,dz\int_0^1 f(t)\,e^{z(x-t)}\,dt \\ &= \frac{1}{4\pi i}\int_{C_{2p}}\frac{e^\zeta}{e^\zeta+1}\,d\zeta\int_0^1 f(t)\,e^{\frac{1}{2}\zeta(x-t)}\,dt \\ &= \frac{1}{2\pi i}\int_{C_{2p}}\frac{e^\zeta}{e^\zeta+1}\,d\zeta\int_0^{\frac{1}{2}} f(2\tau)\,e^{\zeta[(x/2)-\tau]}\,d\tau \\ &= \frac{1}{2\pi i}\int_{C_{2p}}\frac{e^\zeta}{e^\zeta+1}\,d\zeta\int_0^1 g(\tau)\,e^{\zeta[(x/2)-\tau]}\,d\tau. \end{aligned}$$

The poles of $Q_1(z)$ are given by $\zeta = (2\nu + 1)\pi i$, $\nu = 0, \pm 1, \ldots$ so that

$$\sigma_p(x) = \Sigma_{\nu=-p}^{\nu=p-1} e^{(2\nu+1)\pi i x/2} \int_0^1 g(\tau)\, e^{-(2\nu+1)\pi i \tau}\, d\tau$$

$$= e^{\pi i x/2} t_p\left(\frac{x}{2}\right) - e^{(2p+1)\pi i x/2} \int_0^1 g(\tau)\, e^{-(2p+1)\pi i \tau}\, d\tau.$$

Since

$$\int_0^1 g(\tau)\, e^{-(2p+1)\pi i \tau}\, d\tau = o(1)$$

by the Riemann-Lebesgue lemma, the result follows. In order to calculate $J_p(x)$, we shall require estimates for $R(z)$ on C_p. We have

LEMMA 2. *For $z \in C_p$, $R(z) = o(e^{-|\xi|})$.*

Proof. Since $R(z) = Q(z) - Q_1(z)$, we have

$$R(z) = \frac{\int_0^1 \sinh z(1-t)\, \varphi'(t)\, dt}{2\cosh z\left[\cosh z - \int_0^1 \cosh zt\, \varphi'(t)\, dt\right]}. \tag{10}$$

For $z \in C_p^+$,

$$|\sinh z(1-t)| = O(e^{\xi(1-t)}),$$

whence

$$\left|\int_0^1 \sinh z(1-t)\, \varphi'(t)\, dt\right| = o(e^{\xi});$$

by (4), the denominator of $|R(z)|$ exceeds $|Ke^{2z}|$, and so

$$R(z) = o(e^{-\xi})$$

on C_p^+. Since $R(z)$ is an odd function of z, the result follows.

Denote by J_p^+, J_p^-, the contributions to $J_p(x)$ of the arcs C_p^+, C_p^-, respectively. On C_p^+,

$$\int_0^1 f(t)\, e^{-zt}\, dt = o(1), \tag{11}$$

and so, by Lemma 2,

$$J_p^+ = o\left(\int_{C_p+} e^{\xi(x-1)}\, |dz|\right). \tag{12}$$

On C_p^-, we have

$$\int_0^1 f(t)\, e^{z(1-t)}\, dt = o(1),$$

whence, again by Lemma 2,

$$J_p^- = o\left(\int_{C_p^-} e^{\xi x} \mid dz \mid\right). \tag{13}$$

Hence, if x has any assigned value in (0, 1), then by (12) and (13), $J_p(x) \to 0$ as $p \to \infty$. We can then prove

THEOREM 1. *Let* $f \in L(0, 1)$, $\varphi \in AC(0, 1)$ *with* $\varphi(0) = 0$, $\varphi(1) = 1$. *Then,*

$$S_p(x) - e^{\pi i x/2} t_p\left(\frac{x}{2}\right) \to 0$$

as $p \to \infty$ *uniformly in any closed interval interior to* (0, 1).

It is our purpose to examine the behaviour of $S_p(x)$ in [0, 1]. By the imposition of suitable restrictions on f and φ, we can show that $J_p(x) \to 0$ uniformly on the closed interval [0, 1].

2. THEOREMS ON EQUICONVERGENCE

In this section, we prove two theorems which state sufficient conditions for the uniform equiconvergence in [0, 1] of $S_p(x)$ and $e^{\pi i x/2} t_p(x/2)$.

THEOREM 2. *Let* $f \in L(0, 1)$. *Suppose that* φ *is a function such that* $\varphi \in AC(0, 1)$, $\varphi(0) = 0$, $\varphi(1) = 1$, *and* $\varphi' \in BV(0, 1)$. *Then,*

$$S_p(x) - e^{\pi i x/2} t_p\left(\frac{x}{2}\right) \to 0$$

as $p \to \infty$ *uniformly in* [0, 1].

Proof. Since $\varphi' \in BV(0, 1)$, we have

$$\int_0^1 \sinh z(1-t)\, \varphi'(t)\, dt = -\frac{1}{z}\varphi'(1-) + \frac{1}{z}\cosh z \cdot \varphi'(0+) + \frac{1}{z}\int_0^1 \cosh z(1-t)\, d\varphi'(t).$$

Further, on C_p,

$$| \cosh z | = O(e^{|\xi|}), \qquad | \cosh z(1 - t)| = O(e^{|\xi|(1-t)}),$$

and so we obtain

$$R(z) = O\left(\frac{e^{-|\xi|}}{|z|}\right).$$

Hence, by (11),

$$\begin{aligned} J_p^+ &= o\left(\int_{C_p^+} e^{\xi(x-1)} \left|\frac{dz}{z}\right|\right) \\ &= o\left(\int_{C_p^+} e^{\xi(x-1)} \mid d\theta \mid\right) \\ &= o(1), \end{aligned}$$

uniformly for $x \leqslant 1$ as $p \to \infty$.

Similarly,

$$\begin{aligned} J_p^- &= o\left(\int_{C_p^-} e^{\xi x} \left|\frac{dz}{z}\right|\right) \\ &= o(1), \end{aligned}$$

uniformly for $x \geqslant 0$ as $p \to \infty$. Thus, $J_p(x) \to 0$ as $p \to \infty$, uniformly for $0 \leqslant x \leqslant 1$, and the result follows by Lemma 1.

The following result is proved in a similar way:

THEOREM 3. *Let* $\varphi \in AC(0, 1)$, *with* $\varphi(0) = 0$, $\varphi(1) = 1$. *Suppose that* $f \in BV(0, 1)$; *then,*

$$S_p(x) - e^{\pi i x/2} t_p\left(\frac{x}{2}\right) \to 0$$

as $p \to \infty$, *uniformly in* [0, 1].

Proof. The hypothesis that $f \in BV$ allows us to integrate $\int_0^1 e^{z(x-t)} f(t)\, dt$ by parts, and again we obtain an extra factor of $1/z$ in the estimates for J_p^+ and J_p^-. As in Theorem 2, this is sufficient for the result.

3. A THEOREM ON EQUISUMMABILITY

The techniques used in the preceding proofs are not sufficient to ensure uniform equiconvergence in [0, 1] if we weaken the conditions on f and φ

substantially. For example, if we suppose that φ' is bounded instead of being BV, then

$$\left| \int_0^1 \sinh z(1-t)\,\varphi'(t)\,dt \right| \leqslant K \int_0^1 |\sinh z(1-t)|\,dt$$
$$\leqslant \frac{Ke^{|\xi|}}{|\xi|}.$$

The extra factor $1/|\xi|$ is no longer sufficient to ensure that $J_p(x) \to 0$ uniformly, since, C_p close to the imaginary axis, $1/|\xi|$ is large. The troublesome term is the $\cos\theta$ in $\xi = \rho\cos\theta$; however, we can eliminate this by introducing a corrective factor $1 + (z/\rho)^2$. This device was used by Verblunsky [7]. Then, Lemma 1 is no longer applicable, since $Q_1(z)$ is replaced by $Q_1(z)\{1 + (z/\rho)^2\}$. We obtain instead the result:

THEOREM 4. *Let $f \in L(0, 1)$. Suppose that $\varphi \in AC(0, 1)$ with $\varphi(0) = 0$, $\varphi(1) = 1$, and that φ' is bounded. Then, the* C.E.S. *of f is uniformly equisummable* $(C, 1)$ *in* $[0, 1]$ *with $e^{\pi i x/2}$ multiplied by the* F.s. *at $x/2$ of $g(\tau)\,e^{-\pi i \tau}$.*

This is a special case of the following more general theorem.

THEOREM 5. *Let $\varphi \in AC(0, 1)$ with $\varphi(0) = 0$, $\varphi(1) = 1$. Suppose that $f \in L^{p'}(0, 1)$, $\varphi' \in L^{q'}(0, 1)$, where $p' \geqslant 1$, $q' \geqslant 1$ satisfy $1/p' + 1/q' = 1$. Then, the* C.E.S. *of f is uniformly equisummable* $(C, 1)$ *in* $[0, 1]$ *with $e^{\pi i x/2}$ multiplied by the* F.s. *at $x/2$ of $g(\tau)\,e^{-\pi i \tau}$.*

Proof. Write

$$S_p^*(x) = \frac{1}{2\pi i} \int_{C_p} Q(z) \left(1 + \left(\frac{z}{\rho}\right)^2\right) dz \int_0^1 f(t)\, e^{z(x-t)}\, dt,$$

$$\sigma_p^*(x) = \frac{1}{2\pi i} \int_{C_p} Q_1(x) \left(1 + \left(\frac{z}{\rho}\right)^2\right) dz \int_0^1 f(t)\, e^{z(x-t)}\, dt,$$

and

$$K_p(x) = \frac{1}{2\pi i} \int_{C_p} R(z) \left(1 + \left(\frac{z}{\rho}\right)^2\right) dz \int_0^1 f(t)\, e^{z(x-t)}\, dt.$$

As usual, K_p^+, K_p^- are defined in the obvious way. Now,

$$R(z) = O\left(\frac{\left|\int_0^1 \sinh z(1-t)\,\varphi'(t)\,dt\right|}{|\cosh z|^2}\right);$$

let us write

$$\int_0^1 \sinh z(1-t)\,\varphi'(t)\,dt = \left(\int_0^\delta + \int_\delta^1\right)(\sinh z(1-t)\,\varphi'(t)\,dt);$$

then,

$$\begin{aligned}\left|\int_0^\delta \sinh z(1-t)\,\varphi'(t)\,dt\right| &\leqslant \int_0^\delta e^{|\xi|(1-t)}\,|\varphi'(t)|\,dt\\ &\leqslant \left(\int_0^\delta |\varphi'(t)|^{q'}\,dt\right)^{1/q'}\left(\int_0^\delta e^{p'|\xi|(1-t)}\,dt\right)^{1/p'}\\ &= O\left(\epsilon_\delta \frac{e^{|\xi|}}{|\xi|^{1/p'}}\right),\end{aligned}$$

where

$$\epsilon_\delta = \left(\int_0^\delta |\varphi'(t)|^{q'}\,dt\right)^{1/q'}.$$

Next,

$$\begin{aligned}\left|\int_\delta^1 \sinh z(1-t)\,\varphi'(t)\,dt\right| &\leqslant \int_\delta^1 |\varphi'(t)|\,e^{|\xi|(1-t)}\,dt\\ &\leqslant \left(\int_\delta^1 |\varphi'(t)|^{q'}\,dt\right)^{1/q'}\left(\int_\delta^1 e^{p'|\xi|(1-t)}\,dt\right)^{1/p'}\\ &= O\left(\frac{e^{|\xi|(1-\delta)}}{|\xi|^{1/p'}}\right),\end{aligned}$$

and consequently,

$$R(z) = O\left(\frac{\epsilon_\delta}{|\xi|^{1/p'}\,e^{|\xi|}} + \frac{1}{|\xi|^{1/p'}\,e^{|\xi|(1+\delta)}}\right).$$

Now, if $\xi > 0$,

$$\begin{aligned}\left|\int_0^1 f(t)\,e^{-zt}\,dt\right| &\leqslant \left(\int_0^1 |f(t)|^{p'}\,dt\right)^{1/p'}\left(\int_0^1 e^{-q'\xi t}\,dt\right)^{1/q'}\\ &= O(|\xi|^{-1/q'}),\end{aligned}$$

whence

$$\begin{aligned}K_p^+ &= O\left(\int_{C_p^+}\left|1+\left(\frac{z}{\rho}\right)^2\right| e^{\xi x}\left(\frac{\epsilon_\delta}{e^{|\xi|}} + \frac{1}{e^{|\xi|(1+\delta)}}\right)\left|\frac{dz}{\xi}\right|\right)\\ &= O\left(\int_{C_p^+}\{\epsilon_\delta e^{\xi(x-1)} + e^{\xi(x-1-\delta)}\}\,|d\theta|\right).\end{aligned}$$

We have

$$\int_{C_p+} e^{\xi(x-1-\delta)} \mid d\theta \mid = o(1),$$

$$\int_{C_p+} e^{\xi(x-1)} \mid d\theta \mid = O(1),$$

uniformly for $x \leqslant 1$. Hence,

$$K_p^+ = O(\epsilon_\delta)$$

uniformly for $x \leqslant 1$.

If $\xi < 0$,

$$\left| \int_0^1 f(t)\, e^{z(1-t)}\, dt \right| \leqslant \left(\int_0^1 |f(t)|^{p'}\, dt \right)^{1/p'} \left(\int_0^1 e^{-q'|\xi|(1-t)}\, dt \right)^{1/q'}$$
$$= O(|\,\xi\,|^{-1/q'}),$$

so that we obtain

$$K_p^- = O\left(\int_{C_p-} \left| 1 + \left(\frac{z}{\rho}\right)^2 \right| e^{|\xi|(1-x)} \left(\frac{\epsilon_\delta}{e^{|\xi|}} + \frac{1}{e^{|\xi|(1+\delta)}} \right) \left| \frac{dz}{\xi} \right| \right)$$
$$= O\left(\int_{C_p-} \epsilon_\delta e^{-|\xi|x} \mid d\theta \mid + \int_{C_p-} e^{-|\xi|(x+\delta)} \mid d\theta \mid \right).$$

The second integral is uniformly $o(1)$ for $x \geqslant 0$; also,

$$\int_{C_p-} e^{-|\xi|x} \mid d\theta \mid = O(1)$$

uniformly for $x \geqslant 0$, and so

$$K_p^- = O(\epsilon_\delta)$$

uniformly for $x \geqslant 0$.

Since $\epsilon_\delta \to 0$ as $\delta \to 0$, and $K_p(x)$ is independent of δ, it follows that $K_p(x) = o(1)$ uniformly in $[0, 1]$ as $p \to \infty$.

Next,

$$S_p^*(x) = \Sigma_{|z_\nu| < p\pi} \left(1 + \left(\frac{z_\nu}{p\pi} \right)^2 \right) c_\nu e^{z_\nu x} + o(1),$$

where the $o(1)$ term is uniform, and arises from the finite number of non-simple poles of $Q(z)$.

On the other hand, by the argument above,

$$\begin{aligned} S_p^*(x) &= \sigma_p^*(x) + o(1) \\ &= e^{\pi i x/2} \Sigma_{\nu=-p+1}^{\nu=p-1} \left(1 - \frac{(\nu + \frac{1}{2})^2}{p^2} \right) d_\nu e^{\pi \nu i x} + o(1), \end{aligned} \tag{14}$$

where $\Sigma d_\nu e^{\pi\nu ix}$ is the F.s. of $g(\tau)\, e^{-\pi i\tau}$ and the $o(1)$ term is uniform. We may write (14) as

$$e^{\pi ix/2}\Sigma_{\nu=-p+1}^{\nu=p-1}\left(1-\frac{\nu^2}{p^2}\right) d_\nu e^{\pi\nu ix} + o(1),$$

and hence, using (7), we obtain

$$\Sigma_{\nu=-p+1}^{\nu=p-1}\left(1-\frac{\nu^2}{p^2}\right) c_\nu e^{z_\nu x} = e^{\pi ix/2}\Sigma_{\nu=-p+1}^{\nu=p-1}\left(1-\frac{\nu^2}{p^2}\right) d_\nu e^{\pi\nu ix} + o(1), \tag{15}$$

where the $o(1)$ term is uniform.

Define

$$w_0 = c_0 - e^{\pi ix/2}\, d_0\,,$$

$$w_\nu = c_\nu e^{z_\nu x} + c_{-\nu} e^{z_{-\nu} x} - e^{\pi ix/2}\{d_\nu e^{\pi\nu ix} + d_{-\nu} e^{-\pi\nu ix}\};$$

then by (15),

$$\Sigma_{\nu=0}^{\nu=p-1}\left(1-\frac{\nu^2}{p^2}\right) w_\nu = \epsilon_p\,, \tag{16}$$

where $\epsilon_p \to 0$ as $p \to \infty$ uniformly.

We show that this implies $(C, 1)$ summability to zero. To this end, let

$$W_n = \Sigma_{\nu=0}^{\nu=n} w_\nu\,.$$

Then (16) is

$$\Sigma_{\nu=0}^{\nu=p-1}\left(1-\frac{\nu^2}{p^2}\right)(W_\nu - W_{\nu-1}) = \epsilon_p\,,$$

whence

$$\Sigma_{\nu=0}^{\nu=p-1}(2\nu+1)\, W_\nu = p^2\epsilon_p\,.$$

It follows that

$$W_\nu = \tfrac{1}{2}\left(\nu+\tfrac{3}{2}\right)\epsilon_{\nu+1} - \tfrac{1}{2}\left(\nu+\tfrac{1}{2}\right)\epsilon_\nu + o(1),$$

and so

$$\Sigma_{p=0}^{p=n} W_p = \tfrac{1}{2}\left(n+\tfrac{3}{2}\right)\epsilon_{n+1} + o(1).$$

Hence, as $n \to \infty$,

$$\frac{1}{n+1}\Sigma_{p=0}^{p=n} W_p \to 0$$

uniformly, and this proves the theorem.

We have a special case of this theorem when $\varphi \in AC(0, 1)$, $\varphi(0) = 0$, $\varphi(1) = 1$, and f is bounded.

4. Removal of the Condition $\varphi(0) = 0$

We remarked at the beginning that the condition $\varphi(0) = 0$ was included for convenience. We now indicate what happens if $\varphi(0) \neq 0$.

Suppose $\varphi(0) = a$, where a is nonzero. Equation (3) becomes

$$Q(z) = \frac{e^z - a - \int_0^1 e^{zt}\,\varphi'(t)\,dt}{2\left[\cosh z - a - \int_0^1 \cosh zt\,\varphi'(t)\,dt\right]}.$$

The principal part is

$$Q_2(z) = \frac{e^z - a}{2[\cosh z - a]}$$

and the remainder is

$$S(z) = R_1(z) + R_2(z),$$

where

$$R_1(z) = \frac{\int_0^1 \sinh z(1-t)\,\varphi'(t)\,dt}{2[\cosh z - a]\left[\cosh z - a - \int_0^1 \cosh zt\,\varphi'(t)\,dt\right]}$$

and

$$R_2(z) = \frac{a\int_0^1 \sinh zt\,\varphi'(t)\,dt}{2[\cosh z - a]\left[\cosh z - a - \int_0^1 \cosh zt\,\varphi'(t)\,dt\right]}.$$

Choosing C_p so that, for $z \in C_p$, each factor in the denominator exceeds a constant multiple of $|\cosh z|$, we can show, as in Lemma 2, that $R_1(z) = o(e^{-|\xi|})$ on C_p. Hence,

$$\int_{C_p} R_1(z)\,dz \int_0^1 f(t)\,e^{z(x-t)}\,dt = o(1) \tag{17}$$

as $p \to \infty$, for each x in $(0, 1)$.

The denominator of $R_2(z)$ exceeds $Ke^{2|\xi|}$; further,

$$\begin{aligned}\left|\int_0^1 \sinh zt\,\varphi'(t)\,dt\right| &= O\left(\int_0^1 e^{|\xi|t}\,|\varphi'(t)|\,dt\right)\\ &= O\left(e^{|\xi|}\int_0^1 e^{|\xi|(t-1)}\,|\varphi'(t)|\,dt\right)\\ &= o(e^{|\xi|}),\end{aligned}$$

and, hence,

$$\int_{C_p} R_2(z)\, dz \int_0^1 f(t)\, e^{z(x-t)}\, dt = o(1) \tag{18}$$

as $p \to \infty$, for each x in $(0, 1)$.

Uniformity of convergence of the integrals (17) and (18) will occur under the conditions of Theorems 2-5. It remains to consider

$$\frac{1}{2\pi i} \int_{C_p} Q_2(z)\, dz \int_0^1 f(t)\, e^{z(x-t)}\, dt. \tag{19}$$

Define α by setting

$$\cosh \alpha = a.$$

Since $a \neq 0$, α is not a multiple of $\frac{1}{2}\pi i$. If $\operatorname{Im} \alpha$ is not a multiple of $\pi/2$, choose C_p to be $\rho = |z| = (p + \frac{1}{2})\pi$; otherwise, choose C_p to be $|z| = (p + \frac{3}{4})\pi$.

Poles of $Q_2(z)$ occur at $z = 2\nu\pi i \pm \alpha$, and it is easy to verify that all the residues are $\frac{1}{2}$. The number of poles inside C_p is $2p$, for large p.

It follows that (19) is

$$\Sigma_p \tfrac{1}{2} \left\{ \int_0^1 f(t)\, e^{(\alpha+2\pi\nu i)(x-t)}\, dt + \int_0^1 f(t)\, e^{(-\alpha+2\pi\nu i)(x-t)}\, dt \right\},$$

where Σ_p denotes a summation over all poles inside C_p,

$$= \tfrac{1}{2}\, e^{\alpha x} s_p(x, \alpha, f) + \tfrac{1}{2}\, e^{-\alpha x} s_p(x, -\alpha, f) + o(1),$$

where $s_p(x, \alpha, f)$ is the pth partial sum of the F.s. at x of $f(t)\, e^{-\alpha t}$, and contains p terms.

We, therefore, obtain theorems analogous to Theorems 1-5 if we replace

"$e^{\pi i x/2}$ multiplied by the F.s. at $\dfrac{x}{2}$ of $g(\tau)\, e^{-\pi i \tau}$"

by

"the sum of $\frac{1}{2}\, e^{\alpha x}$ multiplied by the F.s. at x of $f(t)\, e^{-\alpha t}$ and $\frac{1}{2}\, e^{-\alpha x}$ multiplied by the F.s. at x of $f(t)\, e^{\alpha t}$."

5. A Bessel-Type Inequality

Suppose that $Q(z)$ has simple poles $\{z_\nu\}$. For large $|\nu|$,

$$z_\nu = (\nu + \tfrac{1}{2})\pi i + o(1),$$

and

$$c_\nu = \lambda_\nu \int_0^1 f(t)\, e^{-z_\nu t}\, dt, \tag{20}$$

where the complex number λ_ν is the residue of $Q(z)$ at z_ν. We prove

THEOREM 6. *Let $\varphi \in AC(0, 1)$ with $\varphi(0) = 0$, $\varphi(1) = 1$. Suppose that $f \in L^2(0, 1)$, and that the numbers $\{c_\nu\}$ are given by* (20). *Then, $\Sigma \mid c_\nu \mid^2 < \infty$.*

Proof. By (2), it is evident that $\lambda_\nu \to \frac{1}{2}$, and so $\mid \lambda_\nu \mid < C$, for all ν. It suffices, therefore, to show that $\Sigma \mid \gamma_\nu \mid^2 < \infty$, where $c_\nu = \lambda_\nu \gamma_\nu$.

The poles $z_\nu = \alpha_\nu + i\beta_\nu$ are such that

$$\alpha_\nu = o(1), \qquad \beta_\nu = (\nu + \tfrac{1}{2})\,\pi + o(1). \tag{21}$$

Writing

$$\begin{aligned} z_\nu &= i(\beta_\nu - i\alpha_\nu) \\ &= -\, i\omega_\nu , \end{aligned}$$

it follows that

$$\mid \operatorname{Im} \omega_\nu \mid < K_1 ,$$

and by (21), that

$$\begin{aligned} \mid \omega_\nu - \bar{\omega}_\mu \mid &= \mid \beta_\mu - \beta_\nu + i(\alpha_\mu + \alpha_\nu)\mid \\ &\geqslant K_2 \mid \nu - \mu \mid , \end{aligned}$$

where K_1, K_2 are constants.

By Lemma 5 of [2], Chap. 1, (or, by Lemma 3 of [6]), there is a constant $A > 0$ such that

$$\int_0^1 \mid \Sigma a_\nu e^{-z_\nu x} \mid^2 dx \leqslant A \Sigma \mid a_\nu \mid^2. \tag{22}$$

Let $\{a_\nu\}$ be any finite set of numbers. Then,

$$\begin{aligned} \mid \Sigma_{\nu=1}^{\nu=p} \gamma_\nu a_\nu \mid &= \left| \int_0^1 f(t)\, \Sigma_{\nu=1}^{\nu=p} a_\nu e^{-z_\nu t}\, dt \right| , \\ &\leqslant \left(\int_0^1 \mid f(t)\mid^2 dt \right)^{1/2} \left(\int_0^1 \mid \Sigma a_\nu e^{-z_\nu t} \mid^2 dt \right)^{1/2} \\ &\leqslant K (\Sigma_{\nu=1}^{\nu=p} \mid a_\nu \mid^2)^{1/2}, \end{aligned} \tag{23}$$

where K is a constant by (22). Now write

$$a_\nu = \frac{\bar{\gamma}_\nu}{\{\Sigma_{\nu=1}^{\nu=p} \mid \gamma_\nu \mid^2\}^{1/2}} \qquad (\nu = 1, 2, \ldots, p);$$

Then, by (23),

$$\Sigma_{\nu=1}^{\nu=p} |\gamma_\nu|^2 \leqslant K^2,$$

and the result follows upon letting $p \to \infty$.

6. A Theorem on The Coefficients

In this section, we shall establish the following formula.

Theorem 7. *Suppose that φ is a positive, nondecreasing normalised BV function on* [0, 1]. *By $\{\delta_\nu\}$ denote the coefficients in the* C.E.S. *of φ with respect to* (2). *Then, writing $\theta_\nu = \delta_\nu + \delta_{-\nu}$ for $\nu = 0, 1, 2,\ldots$,*

$$\int_0^1 x\varphi^2(x)\,dx = \tfrac{1}{2}\,\theta_0{}^2 + \Sigma_{\nu=1}^{\infty}\theta_\nu{}^2. \tag{24}$$

We first prove the following lemma.

Lemma 3. *If φ is positive, nondecreasing, and AC in* [0, 1], *and $\{\delta_\nu\}$ have the significance described, then* (24) *holds.*

Proof. The conditions on φ imply that the roots of

$$\int_0^1 \sin zt\,\varphi(t)\,dt = 0$$

are real and simple ([3]). So we may write the roots of

$$\int_0^1 \sinh zt\,\varphi(t)\,dt = 0$$

as 0, $\pm i\mu_1$, $\pm i\mu_2$,... . We have

$$\begin{aligned}\operatorname{res}_{i\mu_\nu} Q(z) &= \frac{\displaystyle\int_0^1 \varphi(t)\,e^{i\mu_\nu t}\,dt}{2\displaystyle\int_0^1 t\varphi(t)\cos\mu_\nu t\,dt} \\ &= \frac{\displaystyle\int_0^1 \varphi(t)\cos\mu_\nu t\,dt}{2\displaystyle\int_0^1 t\varphi(t)\cos\mu_\nu t\,dt}\end{aligned} \tag{25}$$

and

$$\operatorname{res}_0 Q(z) = \frac{\int_0^1 \varphi(t)\,dt}{2\int_0^1 t\varphi(t)\,dt}. \tag{26}$$

Hence, the C.E.S. of f takes the form

$$\frac{\int_0^1 \varphi(t)\,dt}{2\int_0^1 t\varphi(t)\,dt}\int_0^1 f(t)\,dt + \Sigma_{\nu=1}^{\infty}\frac{\int_0^1 \varphi(t)\cos\mu_\nu t\,dt}{\int_0^1 t\varphi(t)\cos\mu_\nu t\,dt}\int_0^1 f(t)\cos\mu_\nu(x-t)\,dt. \tag{27}$$

Now, if $f \equiv \varphi$, we obtain from (27) the C.E.S. of φ, given by

$$\frac{\theta_0^2}{2\int_0^1 t\varphi(t)\,dt} + \Sigma_{\nu=1}^{\infty}\frac{\theta_\nu^2}{\int_0^1 t\varphi(t)\cos\mu_\nu t\,dt}\cos\mu_\nu x, \tag{28}$$

where

$$\begin{aligned}\theta_\nu &= \int_0^1 \varphi(t)\cos\mu_\nu t\,dt \qquad (\nu = 0, 1, 2,\ldots)\\ &= \delta_\nu + \delta_{-\nu}.\end{aligned}$$

Since φ is AC, we may apply Theorem 3. Then (28) is uniformly equiconvergent in [0, 1] with $e^{\pi i x/2}$ multiplied by the F.s. of $g(\tau)\,e^{-\pi i\tau}$, where

$$g(\tau) = \begin{cases}\varphi(2\tau) & 0 \leqslant \tau \leqslant \frac{1}{2},\\ 0 & \frac{1}{2} < \tau \leqslant 1.\end{cases}$$

We may, therefore, multiply (28) by any function of class $L(0, 1)$ and integrate term by term over [0, 1]. In particular, if we multiply by $x\varphi(x)$, and integrate over [0, 1], we obtain (24). This completes the proof.

Proof of Theorem 7. We suppose that $\varphi(0) = 0$, $\varphi(1) = \varphi(1-) = 1$. We prove that if in Theorem 3 the condition "$\varphi \in AC(0, 1)$" is replaced by "$\varphi \in BV(0, 1)$", and we further assume that f is normalised by $2f(x) = f(x+) + f(x-)$, then there is a subsequence of partial sums of $\Sigma c_\nu e^{z_\nu x}$ which converges boundedly to $f(x)$. Then putting $f = \varphi$, if we multiply by $x\varphi(x)$ and integrate over [0, 1], we obtain (23), as in Lemma 3. The proof depends on arguments similar to those of [5].

We can write

$$Q(z) = Q_1(z) + R(z),$$

where $R(z)$ is given by (10) with $\varphi'(t)\,dt$ replaced by $d\varphi(t)$ in both occurrences. We shall define $\sigma_p(x)$, $J_p(x)$, as in (9). We require information about the contours C_p; let

$$A(z) = z\int_0^1 \sinh zt\,\varphi(t)\,dt = \cosh z - \int_0^1 \cosh zt\,d\varphi(t) \tag{29}$$

There is a constant $C > 0$ such that, if $|\xi| \geqslant C$, then

$$|A(z)| > k\,|\cosh z|\,,$$

where k is constant. Thus, the zeros of $A(z)$ lie in the strip $|\xi| < C$, and the number in the rectangle

$$-C \leqslant \xi \leqslant C, \qquad 2n\pi \leqslant \eta \leqslant (2n+2)\,\pi$$

is bounded, with respect to n, by an argument similar to Lemma 2 of [5].

We can, therefore, find an $\epsilon > 0$ such that if each zero of $A(z)$ is the centre of a disk of radius ϵ, then there is an unbounded increasing sequence $\{\rho_p\}$ such that the circles $C_p = \{|z| = \rho_p\}$ do not pass through any point of the disks. Denote by N_p the greatest integer for which $|z_\nu| < \rho_p$. Then

$$\Sigma_{N_p} c_\nu e^{z_\nu x} = \frac{1}{2\pi i}\int_{C_p} Q(z)\,dz \int_0^1 f(t)\,e^{z(x-t)}\,dt.$$

It can be shown that on C_p

$$|A(z)| > k\,|\cosh z|\,. \tag{30}$$

This is achieved by adapting Verblunsky's "$\delta - \eta$ property" ([5], Section 2); instead of $|H(z)| > \eta > 0$, we have $|H(z)| > k\,|\cosh z|$. Then (30) follows by an argument similar to Lemma 1 of [5]. Hence, on C_p,

$$\frac{e^z}{A(z)} = O(1).$$

Next,

$$\sigma_p(x) = e^{\pi i x/2} t_{N_p}\left(\frac{x}{2}\right),$$

and since $t_r(x/2)$ is the rth partial sum of the F.s. at $x/2$ of $g(\tau)\,e^{-\pi i\tau}$, this converges boundedly in (0, 1) to $f(x)$. Finally, consider $J_p(x)$; we have

$$\int_0^1 f(t)\,e^{z(x-t)}\,dt = \begin{cases} e^{zx}O\left(\dfrac{1}{z}\right) & \text{on} \quad C_p^+, \\[2ex] e^{z(x-1)}O\left(\dfrac{1}{z}\right) & \text{on} \quad C_p^-, \end{cases}$$

since $f \in BV(0, 1)$. Then

$$J_p^+ = O\left(\int_{C_p^+} \left| \frac{R(z)\, e^{zx}}{z}\, dz \right|\right)$$

$$= O\left(\int_{C_p^+} \left| \frac{e^{\xi(x-1)}}{z}\, dz \right|\right),$$

since $R(z) = O(e^{-\xi})$ on C_p^+.

Hence, $J_p^+ \to 0$ boundedly as $p \to \infty$, for $x < 1$. Similarly, $J_p^- \to 0$ boundedly as $p \to \infty$, for $x > 0$, so that $J_p(x) \to 0$ boundedly, for $0 < x < 1$.

There is, therefore, a subsequence of partial sums of the C.E.S. which converges boundedly to $f(x)$ in $(0, 1)$. The theorem now follows as in Lemma 3.

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